BOUNDARY-VALUE PROBLEM IN THE DYNAMICAL THEORY OF ELASTIC ANISOTROPIC MEDIA

V. S. Budaev

UDC 539.3-534.231.1

In the investigation of the propagation of oscillations in elastic anisotropic media one of the most important problems is the elucidation of the features of nonstationary wave fields, not found in isotropic media. From this point of view great significance is attached to problems with concentrated pulsed perturbations, since in a number of cases the solutions to the given problems can be written in finite form in terms of elementary functions, which enables one to conduct a broad investigation of the solutions.

One of the problems, admitting a solution in finite form, is the Lamb problem for a half-plane. The results presented below are a continuation of research [1, 2] on the properties of nonstationary wave fields in anisotropic media. A closed solution to the Lamb problem for an elastic anisotropic half-plane is obtained for the case when the equations of motion under conditions of plane deformation are characterized by four elastic constants.

Examples of the calculation are given for the points on the boundary of the half-plane. The physical consequences are discussed, and the roots of the characteristic equation are investigated in detail.

1. Statement of the Problem and the Solution

We shall consider an elastic anisotropic medium, the equations of motion of which under conditions of plane deformation can be written in the form

$$c_1 \frac{\partial^2 u}{\partial x^2} + c_2 \frac{\partial^2 w}{\partial x \partial z} + c_3 \frac{\partial^2 u}{\partial z^2} = \rho \frac{\partial^2 u}{\partial t^2}, \quad c_3 \frac{\partial^2 w}{\partial x^2} + c_2 \frac{\partial^2 u}{\partial x \partial z} + c_4 \frac{\partial^2 w}{\partial z^2} = \rho \frac{\partial^2 w}{\partial t^2} \tag{1.1}$$

Here c_1, c_2, c_3, and c_4 are coefficients, which are expressed in terms of the elastic constants of the medium

$$c_1 = a_{11},\ c_2 = a_{13} + a_{44},\ c_3 = a_{44},\ c_4 = a_{33}$$

and u and w are the components of the displacement along the x and z axes, and ρ is the density.

Let us consider the boundary-value problem for the half-plane $z \geq 0$, when the boundary conditions at $z = 0$ have the form

$$\sigma_z = -P\delta(x)\ \delta(t), \quad \tau_{zx} = 0 \tag{1.2}$$

with zero initial conditions. The stress-tensor components σ_z and τ_{zx} are written in the form

$$\sigma_z = (c_2 - c_3) \frac{\partial u}{\partial x} + c_4 \frac{\partial w}{\partial z}, \quad \tau_{zx} = c_3 \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \tag{1.3}$$

The functions $\delta(x)$ and $\delta(t)$ are Dirac delta functions.

This is the Lamb problem for an elastic anisotropic half-plane.

In [1] the present problem was discussed for the case when the coefficients in the equations satisfy the conditions

$$\begin{gathered} \gamma > \alpha(1+\beta), \quad \gamma^2 \geqslant 4\alpha\beta, \quad 0 < \alpha < 1, \quad 0 < \beta < 1 \\ [2\beta(1+\alpha) - \gamma(1+\beta)] \geqslant -|\beta - 1|\sqrt{\gamma^2 - 4\alpha\beta} \\ \alpha = c_3/c_1, \ \beta = c_3/c_4, \ \gamma = 1 + \alpha\beta - c_2^2/c_1c_4 \end{gathered} \tag{1.4}$$

Moscow. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 3, pp. 121-125, May-June, 1974. Original article submitted July 3, 1973.

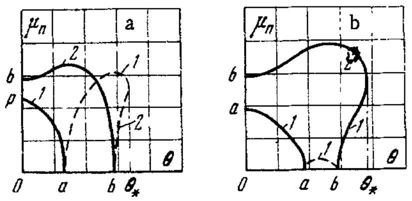

Fig. 1

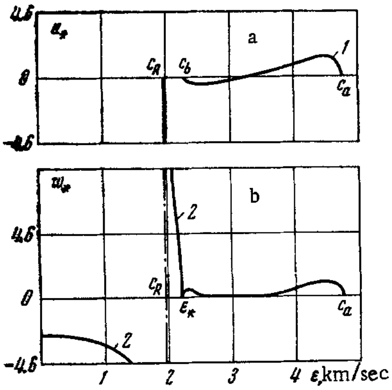

Fig. 2

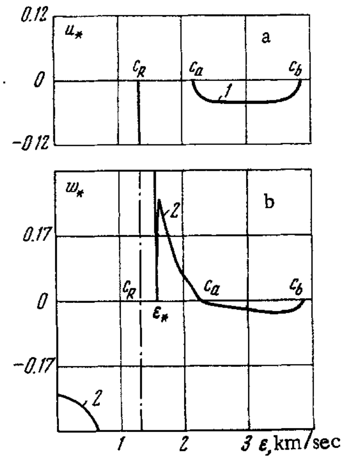

Fig. 3

The solution obtained in [1] is not immediately extended to other cases. To get a solution, valid in all cases, we use a more general method.

Taking Laplace transforms in time, we write the solution to the problem in the form

$$u = -\frac{P}{4\pi^2 c_3}\sum_{n=1}^{2}\int_{c-i\infty}^{c+i\infty} dg \int_{-i\infty}^{i\infty} U_n(\theta)\, e^{gt} e^{-g\xi_n} d\theta$$

$$w = -\frac{P}{4\pi^2 c_3}\sum_{n=1}^{2}\int_{c-i\infty}^{c+i\infty} dg \int_{-i\infty}^{i\infty} W_n(\theta)\, e^{gt} e^{-g\xi_n} d\theta$$

$$\xi_n = \theta x + \mu_n(\theta) z, \quad \mu_n(\theta) = i\theta q_n(\varepsilon), \quad \varepsilon = 1/\theta$$

$$q_n(\varepsilon) = (-M_1 \pm \sqrt{M_1^2 - M_2})^{1/2}, \quad n = 1, 2$$

$$M_1 = \frac{c_2^2 + c_3(\rho\varepsilon^2 - c_3) + c_4(\rho\varepsilon^2 - c_1)}{2c_3c_4}$$

$$M_2 = \frac{(\rho\varepsilon^2 - c_1)(\rho\varepsilon^2 - c_3)}{c_3c_4}$$

$$U_1 = \frac{\theta\alpha_2(\theta)}{F(\theta)}, \quad U_2 = -\frac{\alpha_1(\theta)\Omega_2(\theta)}{F(\theta)}$$

$$W_1 = -\frac{\alpha_2(\theta)\Omega_1(\theta)}{F(\theta)}, \quad W_2 = -\frac{\theta\alpha_1(\theta)}{F(\theta)} \tag{1.5}$$

$$\alpha_1(\theta) = \theta[\mu_1(\theta) + \Omega_1(\theta)], \quad \alpha_2(\theta) = \mu_2(\theta)\Omega_2(\theta) - \theta^2$$

$$\gamma_1(\theta) = b^2p^{-2}[d^2\theta^2 + \mu_1(\theta)\Omega_1(\theta)] - 2\theta^2$$

$$\gamma_2(\theta) = \theta\{b^2p^{-2}[d^2\Omega_2(\theta) - \mu_2(\theta)] - 2\Omega_2(\theta)\}$$

$$F(\theta) = \alpha_1(\theta)\gamma_2(\theta) - \gamma_1(\theta)\alpha_2(\theta)$$

$$\Omega_1(\theta) = -\theta\chi_1(\theta), \quad \Omega_2(\theta) = \theta/\chi_2(\theta)$$

$$\chi_n(\theta) = \frac{c_3\mu_n^2(\theta) + c_1(\theta^2 - a^2)}{\theta c_2\mu_n(\theta)} = \frac{\theta c_2\mu_n(\theta)}{c_4\mu_n^2(\theta) + c_3(\theta^2 - b^2)}$$

$$d^2 = \frac{p^2}{b^2}\left(\frac{b^2 + c^2}{c^2}\right), \quad p^2 = \frac{\rho}{c_4}, \quad c^2 = \frac{\rho}{c_2}, \quad a^2 = \frac{\rho}{c_1}$$

Here $\mu_n(\theta)$ are the roots of the characteristic equation, and θ and g are the integration variables in (1.5).

The functions u and w can be written in the form

$$u = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} E(g)\, e^{gt} dg, \quad w = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Phi(g)\, e^{gt} dg \tag{1.6}$$

$$E(g) = \int_{-i\infty}^{i\infty}\left[\frac{-iP}{2\pi c_3}\sum_{n=1}^{2} U_n(\theta)\right] e^{-g\xi_n} d\theta$$

$$\Phi(g) = \int_{-i\infty}^{i\infty}\left[\frac{-iP}{2\pi c_3}\sum_{n=1}^{2} W_n(\theta)\right] e^{-g\xi_n} d\theta$$

Going from an integration over the imaginary θ axis to an integration along contours L_n, symmetrical with respect to the real axis, along which $\xi_n = t$, i.e.,

$$\mathrm{Re}[\theta x + \mu_n(\theta) z] = t, \quad \mathrm{Im}[\theta x + \mu_n(\theta) z] = 0$$

we obtain

$$E(g) = \sum_{n=1}^{2}\int_0^\infty \left[\frac{-iP}{\pi c_3} U_n(\theta_n)\frac{\partial\theta_n}{\partial t}\right] e^{-gt} dt \tag{1.7}$$

$$\Phi(g) = \sum_{n=1}^{2}\int_0^\infty \left[\frac{-iP}{\pi c_3} W_n(\theta_n)\frac{\partial\theta_n}{\partial t}\right] e^{-gt} dt$$

Applying the inverse Laplace transform (1.6) to (1.7) and separating the real part, we obtain

$$u = u_1 + u_2, \quad w = w_1 + w_2 \tag{1.8}$$

$$u_1 = \frac{P}{\pi c_3}\,\mathrm{Im}\left[\frac{\theta_1\alpha_2(\theta_1)}{F(\theta_1)}\frac{\partial\theta_1}{\partial t}\right]$$

$$u_2 = -\frac{P}{\pi c_3}\,\mathrm{Im}\left[\frac{\alpha_1(\theta_2)\,\Omega_2(\theta_2)}{F(\theta_2)}\,\frac{\partial\theta_2}{\partial t}\right]$$

$$w_1 = -\frac{P}{\pi c_3}\,\mathrm{Im}\left[\frac{\alpha_2(\theta_1)\,\Omega_1(\theta_1)}{F(\theta_1)}\,\frac{\partial\theta_1}{\partial t}\right]$$

$$w_2 = -\frac{P}{\pi c_3}\,\mathrm{Im}\left[\frac{\theta_2\alpha_1(\theta_2)}{F(\theta_2)}\,\frac{\partial\theta_2}{\partial t}\right]$$

The values of θ_n are solutions of the equations

$$t - \theta_n x - \mu_n'(\theta_n) z = 0,\ n = 1,2 \tag{1.9}$$

2. Roots of the Characteristic Equations

When the conditions (1.4) are satisfied, the roots of the characteristic equation for the system (1.1), i.e., the functions $\mu_n(\theta)$, for real θ take only real or purely imaginary values. We shall relate the function $\mu_n(\theta)$ possessing certain properties to the first kind.

Conditions (1.4) are satisfied by a very many anisotropic media and by all isotropic media, so that this kind of medium is very widespread. Of the first kind of medium minerals are especially characteristic, e.g., rock salt, sylvite, feldspar, ice, beryl, sandstone, etc.

Anisotropic media for which conditions (1.4) are not satisfied are also very widespread in nature. In distinction to the first kind of media the metals are most characteristic of these media. A significant fraction of this type is comprised by metals with a cubic lattice and to a lesser degree metals with a hexagonal close-packed structure, which refers to the majority of metals in the second, third, fourth, seventh, and eighth groups of the periodic table. Typical representatives of the latter are beryllium, titanium, cobalt, zinc, rubidium, cadmium, molybdenum, zirconium, tellurium, etc.

The media for which conditions (1.4) are not satisfied are divided into two groups. In one group are the media in which the x axis does not pass through any lacunae. We shall call them media of the second kind. Media of the third kind are those in which the x axis passes through lacunae.

In the case of media of the second and third kind the functions $\mu_n(\theta)$ take complex values for real θ. We shall show that $\mu_n(\theta)$ can take complex values only for $|\theta| > b$, $b = \sqrt{\rho/c_3}$, and, consequently, $q_n(\varepsilon)$ takes complex values only for $|\varepsilon| > c_b$, $c_b = 1/b$.

The points c_1 and c_3 divide the semiaxis $0 \leqslant \rho\varepsilon^2 < \infty$ into three intervals

$$c_1 \leqslant \rho\varepsilon^2 < \infty,\quad c_3 \leqslant \rho\varepsilon^2 < c_1,\ 0 \leqslant \rho\varepsilon^2 < c_3$$

We shall discuss the values of $q_n(\varepsilon)$ in each interval separately.

According to (1.5) we have in the interval $c_1 \leq \rho\varepsilon^2$

$$M_1 > 0,\ M_2 > 0$$

Denoting $k_1 = \rho\varepsilon^2 - c_1$ and $k_2 = \rho\varepsilon^2 - c_3$, we put the expression for $T = M_1^2 - M_2$ in the form

$$T = c_2^4 + 2c_2^2(c_4k_1 + c_3k_2) + (c_4k_1 - c_3k_2)^2$$

whence, considering that $k_1, k_2 > 0$, we obtain that in the given interval $q_n(\varepsilon)$ is purely imaginary, and, consequently,

$$\mu_n(\theta) = i\theta q_n(\varepsilon)\ \text{for}\ |\theta| < a,\ a = \sqrt{\rho / c_1}$$

is real.

In the interval $c_3 \leq \rho\varepsilon^2 < c_1$ we have $M_2 < 0$, so that q_1 is real and q_2 is purely imaginary, independent of the sign of M_1. The function $q_n(\varepsilon)$ can take on complex values only in the interval $0 \leq \rho\varepsilon^2 < c_3$, and $\mu_n(\theta)$ in the interval $|\theta| > b$, $b > a$.

It can be shown that if $\mu_n(\theta)$ is complex at some point $\theta = \theta_1$, then it is complex on the whole interval $\theta_1 \leq \theta < \infty$. The left-most value of θ_1 we denote by θ_*. Crossing the point $\theta = \theta_*$ is associated with a change of sign of the radicant T in the radical $\sqrt{M_1^2 - M_2}$, which for a continuous function is associated with its going to zero at the point $\theta = \theta_*$, from which according to (1.5)

$$\mu_1(\theta_*) = \mu_2(\theta_*) \tag{2.1}$$

From (2.1) it follows that the points $|\theta| = \theta_*$ are branch points of the radical $\sqrt{M_1^2 - M_2}$ in the expression for $\mu_n(\theta)$. Two branch points of the radial $\sqrt{M_1^2 - M_2}$ are real, and, since all branch points are situated symmetrically with respect to the axes, two other branch points lie on the imaginary or real axes. Typical dependences of $\mu_n(\theta)$ for media of the second kind are shown in Fig. 1a for the interval $0 \le \theta < \theta_*$. For media of the third kind the function $\mu_n(\theta)$ also takes complex values on the interval $\theta_* < \theta < \infty$. A representative of these media is copper. In Fig. 1b we show a typical configuration of the curves of $\mu_n(\theta)$ on the real θ axis for media of the form in question. In the present case the function $\mu_n(\theta)$ approaches the point $\theta = \theta_*$, being real. For media of the third kind the points $|\theta| = \theta_*$ are all branch points of the radical $\sqrt{M_1^2 - M_2}$.

The differences in the run of the curves of $\mu_n(\theta)$ cause differences in the run of the displacement curves. When $\mu_n(\theta)$ and $\theta_n(\mu)$ are of the third kind, the x and z coordinate axes pass through lacunae. Near lacuna boundaries the solution behaves the same as near wave fronts.

Besides the functions $\mu_n(\theta)$ the relative values of α and β in comparison with unity exert a great influence on the configuration of the displacement curves, in particular, for the points of the surface.

3. Sample Calculations

We present the dimensionless quantities u_* (curve 1) and w_* (curve 2) in Fig. 2 for zinc (medium of the second kind, $\alpha < 1$, $\beta < 1$) and in Fig. 3 for a model MP medium (second kind, $\alpha > 1$, $\beta < 1$). The quantities u_* and w_* are related to the horizontal and vertical components of the displacement at the points of the boundary

$$u(x, 0, t) = \frac{P}{\pi c_3 t} u_* \left(\frac{x}{t}\right), \quad w(x, 0, t) = \frac{P}{\pi c_3 t} w_* \left(\frac{x}{t}\right) \tag{3.1}$$

Along the abscissa in Figs. 2, 3 the values of $\varepsilon = x/t$ are laid off in km/sec. We shall consider some of the features of the displacement curves. A characteristic feature inherent in the w_* curves in Figs. 2, 3 is the presence of the point ε_* behind the second wave front, where $w_* = 0$. This is associated with the fact that all three media are of the second kind.

In the interval $c_b < \varepsilon < c_a$ the curves of u_* and w_* for zinc do not qualitatively differ from analogous curves for media of the second kind. This is associated with the fact that in both cases $\alpha < 1$, $\beta < 1$. For the material MP the quantity w_* goes to zero at the point $\varepsilon = c_a$ and at the point $\varepsilon = c_b$, which is not possible for media of the third kind. For MP ($\alpha > 1$) the curves on the interval between the wave front ($c_a < \varepsilon < c_b$) have a form fundamentally different from the corresponding first-kind curves. A peculiarity of the material MP is the fact that the Rayleigh-wave velocity is 0.58 of the smallest wave velocity, i.e., for media of the kind in question very low Rayleigh-wave velocities are possible. The indicated features of the surface displacement curves are not possible for isotropic media. [In Fig. 1 curve 1 refers to $\mu_1(\theta)$ and curve 2 to $\mu_2(\theta)$. The imaginary values of the functions $\mu_n(\theta)$ are indicated by the dashed line.]

The author is grateful to S. A. Khristianovich and E. I. Shemyakin for the attention they gave this work.

LITERATURE CITED

1. V. S. Budaev, "On a boundary-value problem in the dynamics of elastic anisotropic media," in: Dynamics of Continuous Media [in Russian], No. 15, Izd. Sib. Otdel. Akad. Nauk (1973).
2. V. S. Budaev, "Propagation of oscillations from a concentrated pulsed-type source in an anisotropic medium," Prikl. Mekhan., 9, No. 2 (1973).